Advanced Mechanics: Final Exam

Semester Ib 2024-2025

January 20th, 2025

Please note the following rules:

- you are not allowed to use the book or the lecture notes, nor other notes or books
- please write your student number on each paper sheet you hand in
- please raise your hand for more paper or to ask a question
- some useful equations are provided on the next page
- it is important that you show your work

Points for each problem:

- Problem 1: *30 points*
- Problem 2: *35 points*
- Problem 3: *20 points*
- Problem 4: *15 points*

Final exam grade = sum of all points.

Useful equations:

- Euler-Lagrange equations:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0\,. \tag{0.1}$$

- Generalised momentum:

$$p_i = \frac{\partial L}{\partial \dot{q}_i}\,. \tag{0.2}$$

- Hamiltonian:

$$H(q_j, p_j; t) = \dot{q}_i p_i - L(q_j, \dot{q}_j; t) \tag{0.3}$$

- Hamilton's equations:

$$\frac{\partial H}{\partial q_j} = -\dot{p}_j\,, \tag{0.4}$$

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \tag{0.5}$$

- Inertia tensor:

$$\mathrm{I_{xx}} = \int \mathrm{dm}\,(y^2 + z^2)\,,\ \mathrm{I_{yy}} = \int \mathrm{dm}\,(x^2 + z^2)\,,\ \mathrm{I_{zz}} = \int \mathrm{dm}\,(x^2 + y^2)\,,$$
$$\mathrm{I_{xy}} = -\int \mathrm{dm}\,x\,y\,, \qquad \mathrm{I_{xz}} = -\int \mathrm{dm}\,x\,z\,, \qquad \mathrm{I_{yz}} = -\int \mathrm{dm}\,y\,z\,. \tag{0.6}$$

- Area of a disk of radius R:

$$\text{Area} = \pi R^2 \tag{0.7}$$

- Area element in polar coordinates (here $x = r\cos\theta$ and $y = r\sin\theta$):

$$\mathrm{dA} = r\,dr\,d\theta \tag{0.8}$$

- Useful integrals:

$$\int d\theta\,\sin^2\theta = \frac{\theta}{2} - \frac{1}{4}\sin(2\theta) \tag{0.9}$$

$$\int d\theta\,\cos^2\theta = \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) \tag{0.10}$$

- Relation between angular momentum and angular velocity of a rigid body:

$$\vec{L} = I \cdot \vec{w}\,, \tag{0.11}$$

- In special relativity, the space-time interval

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

is invariant across reference frames.

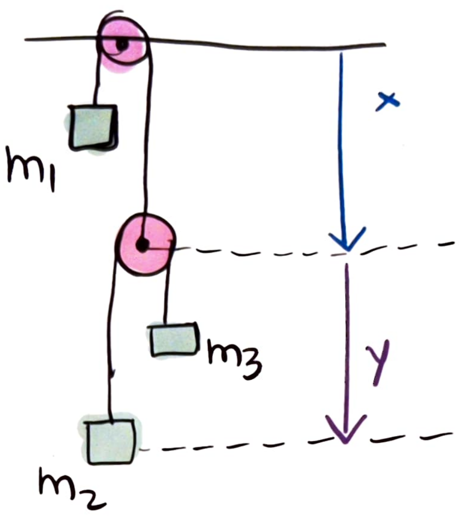

Figure 1.

Problem 1

Consider the system in Fig. 1, consisting of three masses (m_1, m_2 and m_3), subject to gravity, suspended over two identical massless and frictionless pulleys, with two identical massless and inextensible strings. Assuming $m_1 = 4m$, $m_2 = 3m$ and $m_3 = m$

(1a) derive the Lagrangian for the system in terms of the x and y coordinates shown in the figure;
[20 points]

(1b) obtain the Euler-Lagrange equations;
[5 points]

(1c) using the equations found in (1b), solve for the acceleration of mass m_1 when the system is released.
[5 points]

Problem 2

Consider a particle of mass m sitting on a horizontal plane. At time $t = 0$, the plane is raised to an inclination angle α at a rate $\dot{\alpha} = \alpha_0$. As a result, the particle starts moving down the plane. Using the coordinate ρ for the particle's motion as in Fig. 2:

(2a) compute the Lagrangian;
[10 points]

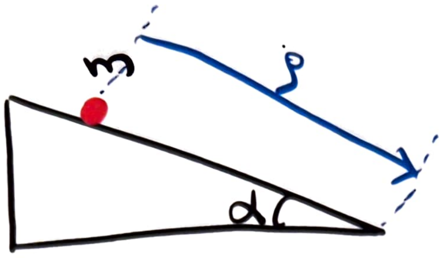

Figure 2.

(2b) obtain the Hamiltonian and Hamilton's equations;
[10 points]

(2c) derive Euler-Lagrange's equation (for ρ) and verify that it is equivalent to Hamilton's equation.
[5 points]

(2d) Solve the equation of motion for ρ, assuming the following initial conditions: $\rho(0) = \rho_0$ and $\dot{\rho}(0) = 0$ (ρ_0 being a constant). *Hint*: the equation of motion for ρ is a linear, inhomogeneous, second-order differential equation, whose solution can be written as the sum $\rho(t) = \rho_{\text{homog}} + \rho_{\text{part}}$, where ρ_{homog} is the general solution to the associated homogeneous equation and ρ_{part} is a particular solution of the full equation, with $\rho_{\text{part}} = C\sin(\alpha_0 t)$ and $C = \text{constant}$.
[10 points]

Problem 3

Consider a disk of radius R and mass M (Fig. 3). For simplicity, assume zero thickness. The disk is located in the (x, y) plane and rotates about the $\hat{z}$ axis (see Fig. 3).

(3a) Which entries of the inertia tensor

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \tag{0.12}$$

are "visibly" equal to zero? *Hint: you should be able to answer this question by quickly looking at the equations for the I_{ij} (i, j=x, y, z) provided in the formula sheet, without doing any actual calculations.*
[5 points]

(3b) Compute I_{xx}, I_{yy} and I_{zz} as a function of the radius R and the mass M of the disk. *Hint: in the formula sheet, you can find some useful equations that will help with this question.*
[10 points]

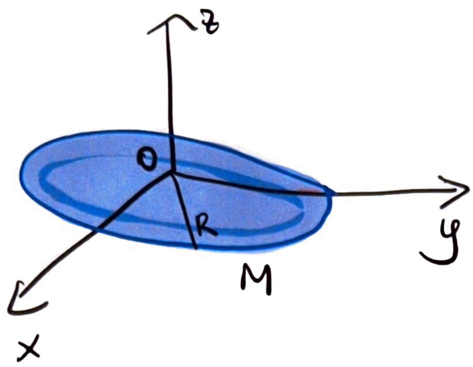

Figure 3.

(3c) Verify that $\hat{z}$ is a principal axis. *Hint: show that if the angular velocity is aligned with $\hat{z}$ the same goes for the angular momentum.*
[5 points]

Problem 4

In this problem, you will study the Lagrangian for a free relativistic particle starting with its action S. S should be a Lorentz invariant quantity. A natural way to define it is with proper time (time measured by a clock in its own rest frame):

$$S = -kc \int_{\tau_1}^{\tau_2} d\tau$$

where c is the speed of light and k is a constant you must determine.

(4a) Show that the Lagrangian for a free relativistic particle of mass m moving in an arbitrary frame with velocity v can be written as

$$L = -mc^2\sqrt{1 - \frac{v^2}{c^2}}$$

Please make sure you explain the steps in your derivation!
Hint: to determine k, evaluate the non-relativistic limit of the Lagrangian (is there any potential U for a free particle?); constant terms are irrelevant in a Lagrangian.
[12 points]

(4b) Derive the generalized momentum ($\vec{v}$ is the generalized velocity).
[3 points]